# Rigidity of pseudo-Anosov flows transverse to R-covered foliations

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**Abstract** – A foliation is **R**-covered if the leaf space in the universal cover is homeomorphic to the real numbers. We show that, up to topological conjugacy, there are at most two pseudo-Anosov flows transverse such a foliation. If there are two, then the foliation is weakly conjugate to the stable foliation of an **R**-covered Anosov flow. The proof uses the universal circle for **R**-covered foliations.

## 1 Introduction

Pseudo-Anosov flows are extremely common amongst 3-manifolds [GK1, Mo2, Fe2, Cal2, Cal3] and they yield important topological and geometrical information about the manifold. For example they imply that the manifold is irreducible and the universal cover is homeorphic to $\mathbf{R}^3$ [Ga-Oe]. There are also relations with the atoroidal property [Fe3]. Finally there are consequences for the large scale geometry of the universal cover when the manifold is atoroidal: In that case it follows that the fundamental group is Gromov hyperbolic [GK2] and in certain cases the dynamics structure of the flow produces a flow ideal boundary to the universal cover which is equivalent to the Gromov boundary and yields many geometric results [Fe7].

As for the existence of pseudo-Anosov flows, it turns out that many classes of Reebless, foliations in atoroidal 3-manifolds admit transverse or almost transverse pseudo-Anosov which are constructed using the foliation structure: 1) fibrations over the circle [Th1], 2) finite depth foliations [Mo2], 3) **R**-covered foliations [Fe2, Cal2] and 4) Foliations with one sided branching [Cal3]. Pseudo-Anosov flows also survive under the majority of Dehn surgeries on closed orbits [GK1], which makes them extremely common. On the other hand there are a few examples of non existence of pseudo-Anosov flows in certain specific manifolds: see [Br] for examples in Seifert fibered spaces and [Ca-Du, Fe5] for examples in hyperbolic manifolds.

In this article we consider the uniqueness question for such flows: Up to topological conjugacy, how many pseudo-Anosov flows are there in a closed 3-manifold? Topological conjugacy means that there is a homeomorphism of the manifold sending orbits to orbits. The less flows there are, the more rigid these flows are and consequently more likely to give information about the manifold. In this generality the question is, at this point, very hard to tackle. Here we start the study of this question and we consider how many pseudo-Anosov flows are there transverse to a given foliation. This is very natural, since as explained above, many pseudo-Anosov flows are constructed from the structure of a given foliation and are transverse to it. We will consider a certain class of foliations called **R**-*covered*: this means that the leaf space in the universal cover is homeomorphic to the set of real numbers [Fe2]. This is the simplest situation with respect to this question. The uniqueness analysis involves a detailed understanding of the topology and geometry of the foliation and flow in this case.

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There are many examples of $\mathbf{R}$-covered foliations: 1) Fibrations over the circle; 2) Many stable and unstable foliations of Anosov flows, which are then called $\mathbf{R}$-covered Anosov flows. These include geodesic flows of hyperbolic surfaces and many examples in hyperbolic 3-manifolds [Fe1]; 3) *Uniform* foliations [Th2]: this means that given any two leaves of the lifted foliation in the universal cover, they are a bounded distance from each other. Obviously the bound depends on the pair of leaves. This is associated with slitherings over the circle [Th2]; 4) Many examples of $\mathbf{R}$-covered but not uniform foliations in hyperbolic 3-manifolds [Cal1].

We should remark that in this article pseudo-Anosov flows include flows without singularities, that is (topological) Anosov flows. On the other hand, we do not allow 1-prong singularities. With 1-prongs almost all control is lost, for example $\mathbf{S}^2 \times \mathbf{S}^1$ has a pseudo-Anosov flow with 1-prongs and the manifold is not even irreducible.

A flow transverse to a foliation is *regulating* if an arbitrary orbit in the universal cover intersects every leaf of the lifted foliation. In particular this implies that the foliation is $\mathbf{R}$-covered. This is strongly related to the atoroidal property: Given an $\mathbf{R}$-covered foliation with a transverse, regulating pseudo-Anosov flow, it follows that either the manifold is atoroidal [Fe3] or it fibers over the circle with fiber a torus and Anosov monodromy. Conversely if the manifold is atoroidal and acylindrical, then there is a regulating, pseudo-Anosov flow transverse to the $\mathbf{R}$-covered foliation [Fe2, Cal2]. So transverse pseudo-Anosov flows are as general as possible in this situation and the uniqueness question is a very natural one in this setting.

There is one case where the uniqueness question for transverse flows is known, which is the simplest case of foliations: a fibration over the circle. It is easy to see that any transverse flow is regulating. Any two transverse flows induce homotopic and hence isotopic monodromies of the fiber $S$. This works even if the flow is not pseudo-Anosov. If the flow is pseudo-Anosov, then the associated monodromy is a pseudo-Anosov homeomorphism of $S$ [Th1]. In particular the fiber cannot be the sphere or the projective plane. If the fiber is Euclidean, then the flow has no singularities and is a topological Anosov flow. In this case it is not hard to prove that there is at most one transverse pseudo-Anosov flow up to conjugacy. Suppose then that the fiber is hyperbolic and therefore the monodromy is pseudo-Anosov with singularities. It is proved in [FLP], exposé 12, that any two homotopic pseudo-Anosov homeomorphisms are in fact conjugate. This implies that the corresponding flows are also topologically conjugate and consequently in this case there is only one transverse pseudo-Anosov flow up to conjugacy.

This result turns out to be very close to what happens in general for $\mathbf{R}$-covered foliations:

**Theorem** – Let $\mathcal{G}$ be an $\mathbf{R}$-covered foliation in $M^3$ closed. Then up to topological conjugacy there is at most one transverse pseudo-Anosov flow which is regulating for $\mathcal{G}$. In addition, up to conjugacy, there is also at most one non regulating transverse pseudo-Anosov flow to $\mathcal{G}$. If there is a transverse pseudo-Anosov flow which is non regulating for $\mathcal{G}$, then the flow has no singular orbits and is a topological Anosov flow. In addition in this case, after a blow down of foliated $I$-bundles, then the resulting foliation $\mathcal{G}'$ is conjugate to either the stable or the unstable foliation of the Anosov flow.

Consequently if $\mathcal{G}$ is not a blow up of the stable/unstable foliation of an $\mathbf{R}$-covered Anosov flow then up to topological conjugacy, there is at most one pseudo-Anosov flow transverse to $\mathcal{G}$. Furthermore there is one such flow if $M$ is atoroidal.

A *foliated $I$-bundle* of $\mathcal{G}$ is an $I$-bundle $V$ embedded in $M$ so that $V$ is a union of leaves of $\mathcal{G}$, which are transverse to the $I$-fibers in $V$. In particular the boundary of $V$ is an union of leaves of $\mathcal{G}$. In general the base of the bundle is not a compact surface. The blow down operation collapses a foliated $I$-bundle onto a single leaf, by collapsing $I$-fibers to points. In the theorem above one may need to do this blow down operation a countable number of times. With reference to the abstract of this article, the phrase $\mathcal{G}$ is weakly conjugate to a foliation $\mathcal{F}$, means that some blow down $\mathcal{G}'$ of $\mathcal{G}$

is topologically conjugate to $\mathcal{F}$.

This theorem generalizes the result for fibrations, because as explained above in that case any transverse flow is regulating.

In order to prove the theorem we split into two cases: the regulating and non regulating situations. The non regulating case was studied in [Fe4] where the second part of the theorem is proved. Here is an outline of that result. The result uses the topological theory of pseudo-Anosov flows, see [Fe4, Fe6]. In the universal cover $\widetilde{M}$ of $M$, the lifted flow has stable and unstable foliations. Since $\mathcal{G}$ is **R**-covered there is only one transverse direction to the lift $\widetilde{\mathcal{G}}$ of the foliation $\mathcal{G}$ to $\widetilde{M}$. After a considerable analysis, this implies that there is only one transverse direction to the stable and unstable foliations of the flow in the universal cover. In particular we show that there are no singularities of the flow $-$ it is a (topological) Anosov flow. In addition we prove that the stable and unstable foliations of the flow $-$ which now are non singular foliations $-$ are **R**-covered foliations. Therefore the flow is an **R**-covered Anosov flow.

The next step is to show that for each leaf of $\widetilde{\mathcal{G}}$ there is a well defined stable (or unstable) leaf in the universal cover associated to it and these two leaves (one stable/unstable and the other a leaf of $\widetilde{\mathcal{G}}$) are a bounded Hausdorff distance from each other. For simplicity assume they are stable leaves. After collapsing foliated $I$-bundles of $\mathcal{G}$, this correspondence between leaves of the stable foliation in the universal cover and leaves of $\widetilde{\mathcal{G}}$ is a bijection. Since the leaf of $\widetilde{\mathcal{G}}$ and the corresponding stable leaf are a bounded Hausdorff distance from each other, there is a map between them which sends a point in one leaf to a point at a bounded distance in the other leaf. As both foliations are **R**-covered then this map is a quasi-isometry. Since leaves of the stable foliation are Gromov hyperbolic [Pl, Su] and any leaf of $\widetilde{\mathcal{G}}$ is quasi-isometric to a stable leaf, it follows that the leaves of $\widetilde{\mathcal{G}}$ are also Gromov hyperbolic. In particular in the non regulating case, there are no parabolic leaves in $\mathcal{G}$. In [Fe4] the analysis was done under the assumption that leaves of $\mathcal{G}$ are Gromov hyperbolic. The argument above shows that this assumption is not necessary. Using a result of Candel [Can], we can assume that the leaves of $\mathcal{G}$ are hyperbolic leaves.

The next step is to show that for each flow line in a fixed leaf of the stable foliation in the universal cover there is a unique geodesic in the corresponding leaf of $\widetilde{\mathcal{G}}$, so that they are a bounded Hausdorff distance from each other. These geodesics in leaves of $\widetilde{\mathcal{G}}$ jointly produce a flow, which projects to a flow in $M$ which is Anosov and whose flow lines are contained in leaves of $\mathcal{G}$. The construction shows that the new Anosov flow is conjugate to the original one and therefore $\mathcal{G}$ is topologically conjugate to the stable foliation of the original Anosov flow.

We remark that it is very easy to construct non regulating examples for certain foliations: let $\mathcal{G}$ be the stable foliation of an **R**-covered Anosov flow $\Psi$. Perturb the flow $\Psi$ slightly along the unstable leaves, to produce a new Anosov flow $\Phi$ which is transverse to $\mathcal{G}$ and non regulating for $\mathcal{G}$.

In this article we consider the regulating situation. The proof is completely different from the non regulating case: in that case the proof was internal to $\widetilde{M}$ $-$ we only used the topology of the pseudo-Anosov flow and showed that stable/unstable leaves and leaves of $\widetilde{\mathcal{G}}$ are basically parallel to each other. Clearly this cannot happen in the regulating situation. In the regulating case we use the asymptotics of the foliation, contracting directions between leaves, the universal circle for foliations and relations of these with the flow. We show that the universal circle of the foliation can be thought of as an ideal boundary for the orbit space of a regulating pseudo-Anosov flow and this can be used to completely determine the flow from outside in $-$ from the universal circle ideal boundary to the universal cover of the manifold in an equivariant way.

The proof of the theorem goes as follows. Let $\Phi$ be transverse and regulating for the foliation $\mathcal{G}$. Suppose first that there is a parabolic leaf. Then we actually show that there has to be a compact leaf which is parabolic. Hence the manifold fibers over the circle with fiber this leaf and the flow is

topologically conjugate to a suspension Anosov flow. In this case there is at most one pseudo-Anosov flow transverse to $\mathcal{G}$, since there cannot be a non regulating transverse pseudo-Anosov flow. This is done in section 2.

In the case that all leaves are Gromov hyperbolic, use Candel's theorem [Can] and assume the leaves are hyperbolic. The orbit space of a pseudo-Anosov flow is the space of orbits in the universal cover. It is always homeomorphic to the plane [Fe-Mo] and the fundamental group of the manifold acts naturally in this orbit space. Given two regulating pseudo-Anosov flows transverse to $\mathcal{G}$ we produce a homeomorphism between the corresponding orbit spaces, which is group equivariant. This is the main step here. Using the foliation $\widetilde{\mathcal{G}}$ which is transverse to each lifted flow, this produces a homeomorphism of the universal cover of the manifold, which takes orbits of one flow to orbits of the other flow and is group equivariant. This produces the conjugacy.

In order to produce the homeomorphism between the orbit spaces, we use in an essential way the universal circle for foliations as introduced by Thurston [Th2, Th3, Th4]. For $\mathbf{R}$-covered foliations, the universal circle is canonically identified to the circle at infinity of any leaf of $\widetilde{\mathcal{G}}$ [Fe2, Cal2]. Notice that the universal circle depends only on the foliation and not on the transverse pseudo-Anosov flow. We first consider only one pseudo-Anosov flow transverse to $\mathcal{G}$. We show that the orbit space of the flow in $\widetilde{M}$ can be compactified with the universal circle of the foliation to produce a closed disk. This is canonically identified with the standard compactification of of any hyperbolic leaf of $\widetilde{\mathcal{G}}$. Here one has to show that the orbit space of the flow in $\widetilde{M}$ and the universal circle of the foliation are compatible with the topology of the leaves of $\widetilde{\mathcal{G}}$ and also that this topology is independent of the particular leaf of $\widetilde{\mathcal{G}}$. To prove this fact, one has to distinguish between uniform and non uniform foliations. Recall that uniform means that any two leaves of $\widetilde{\mathcal{G}}$ are a finite Hausdorff distance from each other $-$ for example fibrations over the circle. The uniform case is simple. The non uniform case requires arguments involving the denseness of contracting directions between leaves, after a possible blow down of foliated $I$-bundles. Using the same ideas we analyse how stable/unstable leaves in the universal cover intersect leaves of $\widetilde{\mathcal{G}}$, particularly with relation to the universal circle. We proved in [Fe6] that for any pseudo-Anosov flow transverse to a foliation with hyperbolic leaves the following happens: given any ray in the intersection of a stable/leaf (in the universal cover) with a leaf of $\widetilde{\mathcal{G}}$, then this ray limits to a single point in the circle at infinity of this leaf of $\widetilde{\mathcal{G}}$. In this article we show if $\mathcal{G}$ is $\mathbf{R}$-covered then given a fixed stable (or unstable leaf) and varying the leaf of $\widetilde{\mathcal{G}}$, then these ideal points in different leaves of $\widetilde{\mathcal{G}}$ follow the identifications prescribed by the universal circle. So clearly the universal circle is intrinsically connected with any regulating, transverse pseudo-Anosov flow. This is done in section 4. These two results are the key tools used in the analysis of the theorem.

The next step is to analyse how an element of the fundamental group acts on the universal circle. If an element of the fundamental group is associated to a closed orbit of the flow, then we show that some power of it acts on the universal circle with a finite even number $\geq 4$ of fixed points and vice versa. This key result depends on the analysis in section 4 and on further properties of the intersections of leaves of $\widetilde{\mathcal{G}}$ and stable/unstable leaves, which is done in section 5.

Finally in section 6 we consider two pseudo-Anosov flows transverse and regulating for $\mathcal{G}$. We first prove that for each lift of a periodic orbit of the first flow, there is a unique periodic orbit of the second flow associated to it. This depends essentially on the study of group actions in section 5. This produces a map between the orbit spaces of the two flows restricted to lifts of closed orbits. The final step is to show that this can be extended to an equivariant homeomorphism between the orbit spaces. This finishes the proof of the theorem.

## 2 The case of parabolic leaves

Leaves of the foliation $\mathcal{G}$ are conformally either spherical, Euclidean or hyperbolic. In this section we rule out the first case and prove the theorem in the second case. We say that a leaf is *parabolic* if it is conformally Euclidean. These terms will be used interchangeably.

The stable and unstable foliations of $\mathcal{G}$ induce 1-dimensional perhaps singular foliations in a leaf $F$ of $\mathcal{G}$. Since there are no 1-prongs in the stable foliation and no centers, then Euler characteristic disallows the existence of spherical leaves.

**Theorem 2.1.** *Let $\mathcal{G}$ be an $\mathbf{R}$-covered foliation transverse to a pseudo-Anosov flow $\Phi$. If $\mathcal{G}$ has a parabolic leaf, then there is a compact leaf $C$ which is parabolic and $M$ fibers over the circle with fiber $C$. In this case the flow is an Anosov flow and is a suspension flow with fiber $C$. Therefore if an $\mathbf{R}$-covered foliation $\mathcal{G}$ has a parabolic leaf, then up to topological conjugacy, there is at most one pseudo-Anosov flow transverse to $\mathcal{G}$.*

*Proof.* We start by proving the first statement. If the pseudo-Anosov flow $\Phi$ is not regulating for $\mathcal{G}$ then as explained in the introduction, the leaves of $\mathcal{G}$ are Gromov hyperbolic and therefore not conformally Euclidean. Therefore $\Phi$ has to be regulating.

We assume first that $M$ is orientable.

Let $L$ be a parabolic leaf of $\mathcal{G}$.

Suppose first that $\mathcal{G}$ has a compact leaf. Since $\mathcal{G}$ is $\mathbf{R}$-covered, it was shown by Goodman and Shields [Go-Sh] that any compact leaf is a fiber of $M$ over the circle. We first want to show that there is a compact leaf which is parabolic. This is not true in general, but it holds for $\mathbf{R}$-covered foliations. If the parabolic leaf $L$ is compact we are done. Suppose then that $L$ is not compact. We want to show that $L$ limits in a compact leaf. In this case let $O$ be the closure of the component of the complement of the compact leaves which contains $L$. This component is a product of a closed surface $C$ times a closed interval and in addition we can assume that $\mathcal{G}$ is transverse to the $I$-fibration in $O$ (see [Fe2]). Identify $C$ with the lower boundary of $O$. Look at the points that $L$ hits in a fixed $I$ fiber $J$. Take the infimum of these points, call it $x$. If $x$ is in the boundary of of $O$ we are done. The foliation in $O$ is determined by its holonomy which is a homomorphism of $\pi_1(C)$ into the group homeomorphisms of $J$. This holonomy has to fix $x$ for otherwise some element would bring $x$ closer to $C$ and hence $L$ would have a point in $J$ lower than $x$. Since the holonomy fixes $x$ then the leaf through $x$ is compact, contrary to assumption that there are no compact leaves in the interior of $O$.

Therefore $L$ limits on a compact leaf $C$. Since $L$ is parabolic, then so is $C$. Since $\Phi$ is regulating for $\mathcal{G}$ then every orbit through $C$ intersects $C$ again, in other words $\Phi$ is a suspension flow and the cross section is an Euclidean surface. In particular $\Phi$ is an Anosov flow. Any two pseudo-Anosov flows transverse to $\mathcal{G}$ will generate suspension flows in $M$ transverse to $C$. As explained in the introduction, any two such flows are topologically conjugate. This finishes the analysis (in the orientable case) when there is a compact leaf.

Suppose now that there is no compact leaf. As proved in proposition 2.6 of [Fe2] there is a unique minimal set $\mathcal{Z}$ in $\mathcal{G}$. Since $L$ must limit in leaves in a minimal set, then there are parabolic leaves in the minimal set, and hence all leaves in the minimal set are parabolic. There are at most countably many components in $M - \mathcal{Z}$ each of which has a closure which is an $I$-bundle over a non compact surface. In addition the flow can be taken to be the $I$-fibration in this closure [Fe2]. Therefore these $I$-bundles can be blown down to leaves to yield a foliation which is still transverse to $\Phi$ and is a minimal foliation. Clearly this happens for any pseudo-Anosov flow transverse to $\mathcal{G}$. Therefore we may assume in this case that $\mathcal{G}$ is minimal.

If all leaves of $\mathcal{G}$ are planes then Rosenberg [Ros] proved that $M$ is homeomorphic to the 3-torus and hence $\pi_1(M)$ has polynomial growth of degree 3. On the other hand a manifold with a pseudo-

Anosov flow has fundamental group with exponential growth [Pl-Th]. Therefore this case cannot happen.

Since $\mathcal{G}$ is minimal and has a parabolic leaf, then all leaves of $\mathcal{G}$ are parabolic. Since $M$ is orientable and $\mathcal{G}$ is transversely orientable (there is a transverse flow), the leaves of $\mathcal{G}$ are either planes, annuli or tori. We took care of the case when there is a toral leaf and also the case when all leaves are planes. Hence there is a leaf, call if $F$, which is an annulus. Since $F$ has polynomial growth, then Plante [Pl] showed that there is a holonomy invariant transverse measure supported in the closure of $F$. Since $F$ is dense, this shows that the support of the measure is all of $M$.

Notice that by Tischler's theorem [Ti] (for the $C^0$ case by Imanishi see [Im]), then $M$ fibers over the circle and $\mathcal{G}$ is approximated arbitrarily close by fibrations. But we need more information.

The next step is to construct an incompressible torus $T$ transverse to $\mathcal{G}$ and foliated by circles. This will take a while. Let $\gamma$ be a simple closed curve in $F$ which is not null homotopic in $F$. Let $B$ be a small closed annulus transverse to $\mathcal{G}$ and with one boundary $\gamma$. Since there is no holonomy in $\mathcal{G}$ the foliation induced by $\mathcal{G}$ in $B$ is a foliation by circles near $\gamma$ and we may assume the other boundary leaf is also a closed curve in a leaf of $\mathcal{G}$. Each of these circles is not null homotopic in its leaf, for otherwise $\gamma$ would be null homotopic in $M$ contradicting Novikov's theorem [No].

Starting from $\gamma$, move along $F$ in a particular side (call it the right side of $B$) until hitting $B$ again. This is possible since leaves of $\mathcal{G}$ are dense in $M$ and $\gamma$ bounds a half non compact annulus in that side. The first time this half annulus hits $B$ again, then it hits $B$ in a closed curve $\gamma_1$ of the induced foliation of $\mathcal{G}$ in $B$. Since $F$ is an annulus then $\gamma$ and $\gamma_1$ bound a closed annulus $A_1$ in $A_1$. We think of $\gamma, \gamma_1$ as oriented, freely homotopic curves in $F$. Let $B_1$ be the closed subannulus of $B$ bounded by $\gamma, \gamma_1$. If $A_1$ approaches $B_1$ from the left side then $\gamma_1$ is freely homotopic to $\gamma$ in $B_1$ (since $M$ is orientable and $\mathcal{G}$ transversely orientable). In this case let $T_1 = A_1 \cup B_1$, which is a two sided torus in $M$.

In the other case $A_1$ approaches $B_1$ from the right side and then $\gamma, \gamma_1$ are freely homotopic to the inverses of each other in $B_1$. In this case $A_1 \cup B_1$ is a one sided Klein bottle. In this case continue along $F$ past $\gamma_1$ until it hits $B_1$ again in a curve $\gamma_2$ (with orientation) $-$ with $A_2$ the annulus in $F$ bounded by $\gamma_1, \gamma_2$. Let $B_2$ be the subannulus of $B_1$ bounded by $\gamma, \gamma_2$ and let $B_3$ be the subannulus of $B_1$ bounded by $\gamma_2, \gamma_1$. If $\gamma_1, \gamma_2$ are freely homotopic in $B_1$, let $T_1 = A_2 \cup B_3$ which is an embedded, two sided torus in $M$. Otherwise let $T_1 = A_1 \cup A_2 \cup B_2$ which is again an embedded two sided torus in $M$.

In any case $T_1$ is a torus obtained from an annulus $A^*$ in $F$ and a transverse annulus contained in $B_1$ foliated by circles. Since the annulus $A^*$ has trivial holonomy, it has a small neighborhood which is product foliated and we can perturb $T_1$ slightly to produce an embedded torus $T$ transverse to $\mathcal{G}$ and foliated by circles. It follows that $T$ is incompressible but we will not need that.

Cut $M$ along $T$ to produce a manifold $M_1$ with 2 boundary tori $U_1, U_2$ and induced 2-dimensional foliation $\mathcal{G}_1$ transverse to the boundary of $M_1$. Since every leaf of $\mathcal{G}$ intersects $B_1$, etc.. then every leaf of $\mathcal{G}_1$ intersects $\partial M_1$. A leaf $E$ of $\mathcal{G}_1$ intersects say $U_1$ in a closed curve $\alpha$ and moving from $\alpha$ in $E$ it has to intersect $\partial M_1$ again. Since the leaves of $\mathcal{G}$ are annuli, then all leaves of $\mathcal{G}_1$ are compact annuli. Since $M_1$ is orientable and has two boundary components, it now follows that $M_1$ is homeomorphic to $S^1 \times V$, where $V$ is a compact annulus. Now $M$ is obtained by glueing $U_1$ to $U_2$ preserving a circle foliation. Hence $M$ is a nilpotent 3-manifold. It follows that $\pi_1(M)$ has polynomial growth, again contradicting the fact that $\pi_1(M)$ has exponential growth [Pl-Th]. So again we conclude that this cannot happen.

We conclude that in this case $\mathcal{G}$ has to have a compact leaf $C$, which is a fiber of a fibration of $M$ over $S^1$ and $\Phi$ is topologically conjugate to a suspension. The result is proved in this case. If $M$ is non orientable then it is doubly covered by an orientable manifold and the result applies to the double cover. Hence again $\mathcal{G}$ has a compact leaf $C$ which is a fiber and the result follows in this case

as well. This finishes the proof the theorem. □

## 3 General facts about R-covered foliations

From now on we may assume that $\mathcal{G}$ has only Gromov hyperbolic leaves. A theorem of Candel [Can] then shows that there is a metric in $M$ so that leaves of $\mathcal{G}$ are hyperbolic surfaces. We assume this is the metric we are using. The following facts concerning **R**-covered foliations are proved in [Fe2, Cal2]. There are two possibilities for $\mathcal{G}$:

- $\mathcal{G}$ is *uniform* $-$ Given any two leaves $L, E$ of $\widetilde{\mathcal{G}}$, then they are a finite Hausdorff distance from each other. This was defined by Thurston [Th2]. If $a$ is the Hausdorff distance between the leaves $L, E$ (which depends on the pair $L, E$), then for any $x$ in $L$ choose $f(x)$ in $E$ so that $d(x, f(x)) \leq a$. This map $f$ is a quasi-isometry between $L$ and $E$ and hence induces a homeomorphism between the corresponding circles at infinity $f : \partial_\infty L \rightarrow \partial_\infty E$. Note that $f$ in general may not even be continuous. However, given the **R**-covered hypothesis, then $f$ is boundedly well defined: any two choices of $f(x)$ are a bounded distance from each other. The bound depends on the pair of leaves. Clearly these identifications between circles at infinity are group equivariant under the action by $\pi_1(M)$. In addition they satisfy a cocycle property: given 3 leaves $L, E, S$ of $\widetilde{\mathcal{G}}$, then the identifications between $\partial_\infty L$ and $\partial_\infty E$ composed with those between $\partial_\infty E$ and $\partial_\infty S$, induce the expected identifications between $\partial_\infty L$ and $\partial_\infty S$. Hence all circles at infinity are identified to a single circle, which is called the *universal circle* of $\mathcal{G}$ or $\widetilde{\mathcal{G}}$ and is denoted by $\mathcal{U}$. By the equivariance property, $\pi_1(M)$ acts on $\mathcal{U}$. The fact to remember here is that given $p$ in $\partial_\infty L$ and $q$ in $\partial_\infty E$, then $p, q$ are associated to the same point of $\mathcal{U}$ if and only if a geodesic ray $r$ in $L$ defining $p$ is a finite Hausdorff distance in $\widetilde{M}$ from a geodesic $r'$ in $E$ defining $q$.

- $\mathcal{G}$ is not uniform. If $\mathcal{G}$ is not a minimal foliation, then it has up to countably many foliated $I$-bundles. One can collapse the $I$-bundles to produce a foliation which is minimal (notice this does not work in the uniform case, for instance when $\mathcal{G}$ is a fibration). If a pseudo-Anosov flow is transverse to $\mathcal{G}$, then one can do the blow down so that the flow is still transverse to the blow down foliation [Fe2]. Sometimes we will assume in this case that $\mathcal{G}$ is minimal. If $\mathcal{G}$ is minimal then the following important fact is proved in [Fe2]: for any $L, E$ leaves of $\widetilde{\mathcal{G}}$, then there is a dense set of contracting directions between them. A *contracting direction* is given by a geodesic $r$ in $L$ so that the distance between $r$ and $E$ converges to 0 as one escapes in $r$. Notice this only depends on the ideal point of $r$ in $\partial_\infty L$ as all such rays are asymptotic because $L$ is the hyperbolic plane. Any such direction produces a *marker* $m$. This is an embedding

$$m : [0, \infty) \times [0, 1] \ \rightarrow \ \widetilde{M}$$

so that for each $s$ in $[0, 1]$ there is a leaf $F_s$ of $\widetilde{\mathcal{G}}$ so that

$$m([0, \infty) \times \{s\}) \ \subset \ F_s$$

is a parametrized geodesic ray in $F_s$. In addition for each $t$ in $[0, +\infty)$ then $m(\{t\} \times I)$ is a transversal to $\widetilde{\mathcal{G}}$ and

$$\forall s_1, s_2 \in I, \quad d(m(t, s_1), m(t, s_2)) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.$$

Hence these geodesics of $F_{s_1}, F_{s_2}$ are asymptotic in $\widetilde{M}$. The contracting directions between $L, E$ induce an identification between dense sets in $\partial_\infty L, \partial_\infty E$ which preserves the circular ordering. This extends to a homemorphism between $\partial_\infty L$ and $\partial_\infty E$. These homeomorphisms are clearly $\pi_1(M)$ equivariant and in addition they satisfy the cocycle property as in the uniform case. Hence as before each circle at infinity is canonically identified to a fixed circle $\mathcal{U}$, the universal circle of $\mathcal{G}$ or $\widetilde{\mathcal{G}}$. Finally $\pi_1(M)$ acts on $\mathcal{U}$.

We now discuss what happens if $\mathcal{G}$ is not uniform and not minimal. This was not discussed in [Fe2] but it is a simple consequence of the analysis of the minimal case as follows: Let $\mathcal{Z}$ be the unique minimal set of $\mathcal{G}$ [Fe2]. Blow down $\mathcal{G}$ to a minimal foliation $\mathcal{G}'$. The analysis above produces the universal circle $\mathcal{U}'$ for $\mathcal{G}'$. Let $\delta : M \to M$ be the blow down map sending leaves of $\mathcal{G}$ to leaves of $\mathcal{G}'$ and homotopic to the identity. Lift the homotopy to produce a lift $\widetilde{\delta}$ of $\delta$, which is a homeomorphism of $\widetilde{M}$. For any $A, B$ leaves of $\widetilde{\mathcal{G}}'$, there are $F, E$ leaves in $\widetilde{\mathcal{Z}}$ so that $A, B$ are between $F, E$. Let $F' = \widetilde{\delta}(F)$, $E' = \widetilde{\delta}(E)$. Then in $\widetilde{\mathcal{G}}'$ there is a dense set of contracting directions between $F'$ and $E'$. For any such there is a ray $r'$ in $F'$ asymptotic to a ray $l'$ in $E'$. Under the blow up map, this produces corresponding rays in $F, E$: a ray $r$ in $F$ which is a bounded distance from a ray $l$ in $E$. By the $\mathbf{R}$-covered property, the ideal point of the ray $l$ is the unique direction for which there is a ray a bounded distance from $r$ in $\widetilde{M}$. This provides an identification between dense sets in $\partial_\infty F$ and $\partial_\infty E$. This is equivariant and satisfies the cocycle property. This can be extended to a group equivariant homeomorphism between $\partial_\infty F$ and $\partial_\infty E$. This produces the universal circle in this case.

Calegari [Cal1] produced many examples of $\mathbf{R}$-covered, non uniform foliations in closed, hyperbolic 3-manifolds.

## 4 Intersections between leaves of $\widetilde{\mathcal{G}}$ and pseudo-Anosov foliations

Let $\Phi$ be a pseudo-Anosov flow in $M^3$ closed. Background on pseudo-Anosov flows can be found in [Mo1, Fe6]. Here we always assume that there are no 1-prong singular orbits. The universal cover of $M$ is denoted by $\widetilde{M}$. Let $\mathcal{F}^s, \mathcal{F}^u$ be the stable/unstable foliations of $\Phi$ and $\widetilde{\Phi}, \widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ the lifts to the universal cover of $\Phi, \mathcal{F}^s, \mathcal{F}^u$ respectively. Given $z$ in $\widetilde{M}$ let $\widetilde{W}^s(z)$ be the stable leaf containing $z$ and similarly define $\widetilde{W}^u(z)$. Our assumption is that $\Phi$ is transverse to the foliation $\mathcal{G}$ and is regulating for $\mathcal{G}$. Therefore given any leaf $L$ of $\widetilde{\mathcal{G}}$, the foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ are transverse to $L$ and they induce 1-dimensional singular foliations $\widetilde{\mathcal{F}}^s_L, \widetilde{\mathcal{F}}^u_L$ in $L$. We are in the case that leaves of $\widetilde{\mathcal{G}}$ are isometric to the hyperbolic plane.

One key fact to be used here is that we proved in [Fe6] that each ray of a leaf of $\widetilde{\mathcal{F}}^s_L$ or $\widetilde{\mathcal{F}}^u_L$ accumulates in a single point of $\partial_\infty L$. This works even if $\mathcal{G}$ is not $\mathbf{R}$-covered.

A convention that will be used throughout the article is the following: the group $\pi_1(M)$ acts on several objects: the universal cover $\widetilde{M}$, the orbit space $\mathcal{O}$, the universal circle $\mathcal{U}$, the foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u, \mathcal{O}^s, \mathcal{O}^u$, etc.. If $g$ is an element of $\pi_1(M)$ we still use the same $g$ to denote the induced actions on all these spaces $\widetilde{M}, \mathcal{O}, \mathcal{U}, \widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u, \mathcal{O}^s, \mathcal{O}^u$, etc..

**Lemma 4.1.** *Suppose that a pseudo-Anosov flow $\Phi$ is regulating for an $\mathbf{R}$-covered foliation $\mathcal{G}$. Then the stable and unstable foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ have Hausdorff leaf space. Therefore for any leaf $L$ of $\widetilde{\mathcal{G}}$, the leaves of the one dimensional foliations $\widetilde{\mathcal{F}}^s_L, \widetilde{\mathcal{F}}^u_L$ are uniform quasigeodesics in $L$.*

*Proof.* This is stronger than the fact that rays in these leaves limit to single points in $\partial_\infty L$. If we suppose on the contrary that (say) $\widetilde{\mathcal{F}}^s$ does not have Hausdorff leaf space, then there are closed orbits $\alpha, \beta$ of $\Phi$ (maybe with multiplicity), so that they are freely homotopic to the inverse of each other, see [Fe6]. Lift them coherently to orbits $\widetilde{\alpha}, \widetilde{\beta}$ of $\widetilde{\Phi}$. Since $\Phi$ is regulating for $\mathcal{G}$, then both $\widetilde{\alpha}$ and $\widetilde{\beta}$ intersect every leaf of $\widetilde{\mathcal{G}}$.

Let $g$ in $\pi_1(M)$ non trivial with $g$ leaving $\widetilde{\alpha}$ invariant and sending points in $\widetilde{\alpha}$ forward (in terms of the flow parameter). Therefore $g$ acts in an increasing way in the leaf space of $\widetilde{\mathcal{G}}$. By the free homotopy, $g$ also leaves $\widetilde{\beta}$ invariant and $g$ acts decreasingly in $\widetilde{\beta}$, hence also in the leaf space of $\widetilde{\mathcal{G}}$. This is a contradiction.

Hence the leaf spaces of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ are Hausdorff. As proved in proposition 6.11 of [Fe6] this implies that for any $L$ in $\widetilde{\mathcal{G}}$, then all leaves of $\widetilde{\mathcal{F}}^s_L, \widetilde{\mathcal{F}}^u_L$ are uniform quasigeodesics in $L$. The bounds are independent of the leaf of $\widetilde{\mathcal{F}}^s_L, \widetilde{\mathcal{F}}^u_L$ in $L$ and also of the leaf $L$ of $\widetilde{\mathcal{G}}$. For non singular leaves, this implies that any such leaf is a bounded distance (in the hyperbolic metric of $L$) from a minimal geodesic in $L$. For singular $p$-prong leaves of $\widetilde{\mathcal{F}}^s_L, \widetilde{\mathcal{F}}^u_L$ the same is true for any properly embedded copy of $\mathbf{R}$ in such leaves. $\square$

In this section we want to show that the asypmtotic behavior of leaves of $\widetilde{\mathcal{F}}^s_L, \widetilde{\mathcal{F}}^u_L$ is coherent with the identifications prescribed by the universal circle.

Let $\mathcal{H}$ be the leaf space of $\widetilde{\mathcal{G}}$, which is homeomorphic to the set of real numbers. Let $\pi : \widetilde{M} \rightarrow M$ be the universal covering map.

Let $r$ be ray of $\widetilde{\mathcal{F}}^s_L$ (or $\widetilde{\mathcal{F}}^u_L$) starting at a point $p$ in $L$. In general this is not a geodesic ray in $L$. Let $E$ be any leaf of $\widetilde{\mathcal{G}}$. Since $\Phi$ is regulating, then $\widetilde{\Phi}_{\mathbf{R}}(p)$ intersects $E$ and the same is true for any point $q$ in $r$. The intersection of $\widetilde{\Phi}_{\mathbf{R}}(r)$ and $E$ is a ray of $\widetilde{\mathcal{F}}^s_E$ $-$ again because of the regulating condition. This ray also defines an unique ideal point in $\partial_\infty E$. Since $\partial_\infty E$ is canonically identified with the universal circle $\mathcal{U}$ this defines a map

$$f_r : \mathcal{H} \ \rightarrow \ \mathcal{U}$$

$$f_r(E) \ = \ \{ \text{ equivalence class in } \ \mathcal{U} \ \text{ of the ideal point in } \ \partial_\infty E \ \text{ of the ray } \ (\widetilde{\Phi}_{\mathbf{R}}(r) \cap E) \ \}$$

**Proposition 4.2.** *For any $L$ in $\widetilde{\mathcal{G}}$ and any $r$ ray of $\widetilde{\mathcal{F}}^s_L$ or $\widetilde{\mathcal{F}}^u_L$, then $f_r : \mathcal{H} \rightarrow \mathcal{U}$ is a constant map.*

*Proof.* The proof depends on whether $\mathcal{G}$ is uniform or not.

Case 1 $-$ $\mathcal{G}$ is uniform.

Claim $-$ If $\mathcal{G}$ is uniform and $\Phi$ is transverse and regulating for $\mathcal{G}$, then for any $S, E$ leaves of $\widetilde{\mathcal{G}}$, there is a bound on the length of flow lines from $S$ to $E$. The bound depends on the pair $S, E$.

Otherwise we find $p_i$ in $S$ with $\widetilde{\Phi}_{t_i}(p_i)$ in $E$ and $t_i$ converging to (say) infinity. Up to subsequence assume that $\pi(p_i)$ converges to a point $p$ in $M$. Take covering translations $g_i$ in $\pi_1(M)$ with $g_i(p_i)$ converging to $p_0$. For each $i$ take $q_i$ in $g_i(E)$ with $d(q_i, g_i(p_i)) < a$ for fixed $a$. This uses the uniform property. Up to subsequence assume that $q_i$ converges and hence $g_i(E)$ converges to a leaf $E_0$. The orbit of $\widetilde{\Phi}$ through $p_0$ intersects $E_0$, since the flow is regulating. Hence there is $t_0$ with $\widetilde{\Phi}_{t_0}(p_0)$ in $E_0$. By continuity of flow lines of $\widetilde{\Phi}$, then for any $z$ in $\widetilde{M}$ near $p_0$ and $G$ leaf of $\widetilde{\mathcal{G}}$ near $E_0$, then there is $t$ near $t_0$ so that $\widetilde{\Phi}_t(z)$ is in $G$. But $\widetilde{\Phi}_{t_i}(g_i(p_i))$ is in $g_i(E)$, which is a leaf near $E_0$ and $t_i$ converges to infinity, contradiction. This proves the claim. Notice that it is not necessary for $\Phi$ to be pseudo-Anosov in this claim, just that it is regulating.

Let $l$ be the geodesic ray in $L$ with starting point $p$ and a finite Hausdorff distance (in $L$) from $r$. By the above $\widetilde{\Phi}_{\mathbf{R}}(r)$ intersects $E$ in a ray $r'$ of $\widetilde{\mathcal{F}}^s_E$ which is a bounded distance from $r$ in $\widetilde{M}$. The ray $r'$ is also a uniform quasigeodesic ray in $E$, hence $r'$ is a bounded distance in $E$ from a geodesic ray $l'$. Then $l, l'$ are a finite distance from each other in $\widetilde{M}$. The definition of the universal circle in the uniform case implies that $r, r'$ define the same point in $\mathcal{U}$. This establishes this case.

Case 2 $-$ $\mathcal{G}$ is not uniform.

In this case, first assume that $\mathcal{G}$ is minimal. Therefore between any two leaves of $\widetilde{\mathcal{G}}$, there is a dense set of contracting directions. The proof will be done by contradiction. Let $r$ be a ray of a leaf of $\widetilde{\mathcal{F}}^s_L$ for some $L$ in $\widetilde{\mathcal{G}}$ with initial point $p$. Let $a$ be the ideal point of $r$ in $\partial_\infty L$. Suppose that for some $E$ leaf of $\widetilde{\mathcal{G}}$, then

$$r' \ = \ \widetilde{\Phi}_{\mathbf{R}}(r) \cap E \quad \text{defines a distinct point in} \quad \mathcal{U}$$

Let $b$ be the point in $\partial_\infty L$ identified to the ideal point of $r'$ in $\partial_\infty E$, by the universal circle identification. Hence $a, b$ are different. By density of contracting directions between $L$ and $E$, there are points $c, d$ in $\partial_\infty L$ which separate $a$ from $b$ in $\partial_\infty L$ and so that $c, d$ correspond to contracting directions between $L$ and $E$. Let $m_1, m_2$ be markers between $L$ and $E$ associated to the contracting directions $c, d$ respectively. Let $B_i = Image(m_i)$ and let $C$ be the union of the points in $\widetilde{M}$ contained in leaves intersecting the markers $m_1, m_2$. Removing initial pieces if necessary we may assume that $B_1, B_2$ are disjoint. Since $m_i(\{t\} \times I)$ is a very small transverse arc if $t$ is big enough, we can also assume the following: if $z$ is in $B_1$ or $B_2$ then $\widetilde{\Phi}_{\mathbf{R}}(z)$ will intersect any leaf $S$ in $C$ near $z$, producing a small transversal from $L$ to $E$ passing through $z$. For each leaf $S$ of $\widetilde{\mathcal{G}}$ intersected by the markers, let

$$r_S \ = \ \text{geodesic arc in} \ \ S \ \ \text{joining the endpoints of} \ \ Image(m_1) \cap S \ \ \text{and} \ \ Image(m_2) \cap S.$$

Let $A$ be the union of the $r_S$ for such $S$. This is topologically a rectangle with the bottom in $L$ the top in $E$ and the sides transversals from $L$ to $E$. Then $A \cup B_1 \cup B_2$ separates $C$ into 2 components $C_1, C_2$. Since $\{a, b\}$ is disjoint from $\{c, d\}$ the ray $r$ does not accumulate on $c$ or $d$ in $\partial_\infty L$. Hence starting with a smaller ray $r$ if necessary we may assume also that $r, r'$ are disjoint from $B_i$ and far away from it. In particular the flow line through any point of $r$ will not intersect $B_i$, since points in $B_i$ are in very short transversals from $L$ to $E$.

By renaming $C_1, C_2$ we may assume that $r$ is in $C_1$ and $r'$ is in $C_2$. For each $z$ in $r$ it is in $C_1$, then the flow line through $z$ intersects $E$ in $r'$ which is in $C_2$. Therefore this flow line has to intersect $A \cup B_1 \cup B_2$. The above remarks imply that this flow line cannot intersect either $B_1$ or $B_2$. Hence this flow line must intersect $A$. Since $A$ is compact we can choose $z_i$ in $r$ escaping in $r$ so that $\widetilde{\Phi}_{\mathbf{R}}(z_i)$ intersects $A$ in

$$q_i \ = \ \widetilde{\Phi}_{t_i}(z_i) \quad \text{and} \quad q_i \to q \in A$$

Since $z_i$ escapes in $r$, it follows that $t_i$ converges to infinity. By the regulating property of $\Phi$, the orbit through $q$ intersects $L$. Hence nearby orbits intersect $L$ in bounded time, contradicting that $t_i$ converges to infinity.

This contradiction shows that $r'$ has to define the same point in $\mathcal{U}$ that $r$ does. This finishes the proof when $\mathcal{G}$ is minimal.

If $\mathcal{G}$ is not minimal, then first blow down $\mathcal{G}$ to a minimal foliation $\mathcal{G}'$. We can assume that $\Phi$ is still transverse to $\mathcal{G}'$. Now use the proof for $\mathcal{G}'$ as above. The walls $A \cup B_1 \cup B_2$ for $\widetilde{\mathcal{G}}'$ pull back to walls for $\widetilde{\mathcal{G}}$. Because the foliation $\mathcal{G}$ is a blow up of $\mathcal{G}'$ and $\Phi$ is transverse to both of them, it follows that flowlines of $\widetilde{\Phi}$ cannot cross the two ends of the pullback walls and if necessary can only cross the compact part of these walls. Therefore the same arguments as above prove the result in this case. This finishes the proof of the proposition. $\square$

For a leaf $F$ of $\widetilde{\mathcal{G}}$ we consider $F \cup \partial_\infty F$ as the canonical compactification of $F$ as a hyperbolic plane. Given any two leaves $F, E$ in $\widetilde{\mathcal{G}}$, then using the universal circle analysis there is a homeomorphism between $\partial_\infty F$ and $\partial_\infty E$. In addition if a flow $\Phi$ is regulating for $\mathcal{G}$ then there is also a homeomorphism between $F, E$ by moving along flow lines. We next show that these are compatible:

**Proposition 4.3.** *Given $F, E$ in $\widetilde{\mathcal{G}}$ consider the map $g$ from $F \cup \partial_\infty F$ to $E \cup \partial_\infty E$ defined by: if $x$ is in $F$ then move along the flow line of $\widetilde{\Phi}$ through $x$ until it hits $E$. The intersection point is $g(x)$. If $x$ is in $\partial_\infty F$, let $g(x)$ be the point in $\partial_\infty E$ associated to $x$ by the universal circle identification. Then $g$ is a homeomorphism. In addition these homeomorphisms are group equivariant and satisfy the cocycle condition.*

*Proof.* The map $g$ is a bijection. We only need to show that it is continuous, since the inverse is a map of the same type. The equivariance and cocycle properties follow immediately from the same properties for flowlines and identifications induced by the universal circle.

We now prove continuity of $g$: This is very similar to the previous proposition and we will use the setup of that proposition. The first possibility is that $\mathcal{G}$ is uniform. Then as seen in the previous proposition the map $g : F \to E$ is a quasi-isometry and it induces a homeomorphism $g^*$ from $F \cup \partial_\infty F$ to $E \cup \partial_\infty E$. The image of an ideal point $p$ in $\partial_\infty F$ is determined by the ideal point of $g(r)$ where $r$ is a geodesic ray in $F$ with ideal point $p$. But $g(r)$ is a bounded distance from $r$ in $\widetilde{M}$ and this is exactly the identification associated to the universal circle.

Suppose now that $\mathcal{G}$ is not uniform. Suppose first that $\mathcal{G}$ is minimal. We know that $g$ restricted to both $F$ and $\partial_\infty F$ are homeomorphisms. Since $F$ is open in $F \cup \partial_\infty F$ all we need to do is to show that $g$ is continuous in $\partial_\infty F$. Let $a$ in $\partial_\infty F$ and $(a_i)$ converging to $a$ in $F \cup \partial_\infty F$, so we may assume that $a_i$ is in $F$. Suppose by way of contradiction that $g(a_i)$ converges to $g(b)$ where $b$ is not $a$. Choose $c, d$ in $\partial_\infty F$ which separate $a, b$ in $\partial_\infty F$. Then construct the wall $A \cup B_1 \cup B_2$ as in the previous proposition. The flow lines from $a_i$ to $g(a_i)$ have to intersect this wall in a compact set, contradiction as in the previous proposition. This finishes the proof if $\mathcal{G}$ is minimal.

If $\mathcal{G}$ is not minimal, then use the same arguments as in the end of the previous proposition to deal with this case. □

This proposition allows us to put a topology in $\mathcal{O} \cup \mathcal{U}$ as follows: Consider any leaf $L$ of $\widetilde{\mathcal{G}}$. There are homeomorphisms between $L$ and $\mathcal{O}$ and $\partial_\infty L$ and $\mathcal{U}$. The combined map induces a topology in $\mathcal{O} \cup \mathcal{U}$. The previous proposition shows that this topology is independent of the leaf $L$ we start with. In addition covering translations induce homeomorphisms of $\mathcal{O} \cup \mathcal{U}$ − this is because if $L$ is in $\widetilde{\mathcal{G}}$ and $g$ in $\pi_1(M)$ then $g$ is a homeomorphism from $L \cup \partial_\infty L$ to $(g(L) \cup \partial_\infty g(L))$, both of which are homeomorphic to $\mathcal{O} \cup \mathcal{U}$. We think of this as an action on $\mathcal{O} \cup \mathcal{U}$. Given $g$ in $\pi_1(M)$, then the notation $g$ will also denote the induced map in $\mathcal{O} \cup \mathcal{U}$. The analysis above makes it clear that $g$ in $\pi_1(M)$ acts as an orientation preserving way on $\mathcal{O}$ if and only if it acts as an orientation preserving way on $\mathcal{U}$.

## 5 Action of elements of $\pi_1(M)$

The main purpose of this section is to analyse how elements of $\pi_1(M)$ act on $\mathcal{U}$. We first need a couple of auxiliary results. Here is some notation/terminology which will be used in the sequel. Let $\mathcal{O}$ be the orbit space of the lifted flow $\widetilde{\Phi}$. The space $\mathcal{O}$ is always homeomorphic to the plane [Fe-Mo]. The foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ induce 1-dimensional, possibly singular foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O}$. The only possible singularities are of $p$-prong type. A point $x$ in $\mathcal{O}$ is called *periodic* if there is $g \neq id$ in $\pi_1(M)$ with $g(x) = x$. Let

$$\Theta : \ \widetilde{M} \ \rightarrow \ \mathcal{O} \quad \text{be the projection map}$$

An orbit $\alpha$ of $\widetilde{\Phi}$ is periodic if $\Theta(\alpha)$ is periodic. A *line leaf* of $\widetilde{\mathcal{F}}^s_L$ is a properly embedded copy of $\mathbf{R}$ in a leaf of $\widetilde{\mathcal{F}}^s_L$ of a leaf $L$ of $\widetilde{\mathcal{G}}$ so that: if $l$ is in a singular leaf $r$ of $\widetilde{\mathcal{F}}^s_L$, then $r - l$ does not have prongs of $r - l$ on both sides of $l$ in $L$. A singular leaf with a p-prong singularity has $p$ lines leaves. Consecutive

line leaves intersect in a ray of $\widetilde{\mathcal{F}}^s_L$. Non singular leaves are line leaves themselves. Similarly one defines line leaves for $\widetilde{\mathcal{F}}^u_L, \mathcal{O}^s, \mathcal{O}^u, \widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$. Given $z$ in $\widetilde{M}$ let $\widetilde{W}^s(z)$ be the stable leaf containing $z$. The *sectors* of $\widetilde{W}^s(z)$ are the connected components of $\widetilde{M} - \widetilde{W}^s(z)$.

**Lemma 5.1.** *Let $\Phi$ be regulating for $\mathcal{G}$ which is $\mathbf{R}$-covered with hyperbolic leaves. Let $l_i$ be line leaves of $\widetilde{\mathcal{F}}^s_{L_i}$ where $L_i$ are leaves of $\widetilde{\mathcal{G}}$. Suppose that there are $p_i$ in $l_i$ so that $p_i$ converges in $\widetilde{M}$ to a point $p$ in a leaf $L$ of $\widetilde{\mathcal{G}}$. If $\widetilde{W}^s(p)$ is singular assume that all $p_i$ are in the closure of a sector of $\widetilde{W}^s(p)$. Then there is a line leaf $l$ of $\widetilde{\mathcal{F}}^s_L$ with $p$ in $l$ and $l_i$ converging to $l$ in the geometric topology of $\widetilde{M}$. In addition if $s_i$ are the geodesics in $L_i$ a bounded distance from $l_i$ in $L_i$ and $s$ is the geodesic a bounded distance from $l$ in $L$ then $s_i$ converges to $s$ in the geometric topology of $\widetilde{M}$.*

*Proof.* We first prove the statement about $l_i$ and $l$. Geometric convergence means that if $z$ is in $l$ then there are $z_i$ in $l_i$ with the sequence $(z_i)$ converging to $z$ and in addition if $z_{i_k}$ is in $l_{i_k}$ and $(z_{i_k})$ converges to $w$ in $\widetilde{M}$ then $w$ is in $l$.

Since the flow $\Phi$ is regulating for $\mathcal{G}$, then $l_i$ flows into line leaves $r_i$ of $\widetilde{\mathcal{F}}^s_L$. The points $p_i$ flow to $q_i$ in $L$ and clearly $q_i$ converges to $p$. Hence there is a line leaf $l$ of $\widetilde{\mathcal{F}}^s_L$ through $p$, so that any point $z$ in $l$ is the limit of a sequence $(z'_i)$ with $z'_i$ in $r_i$. If $\widetilde{W}^s(p)$ is singular, this uses the fact that the $p_i$ are all in the closure of a sector of $\widetilde{W}^s(p)$. Otherwise it could easily be that different subsequences of $r_i$ converge to distinct line leaves of $\widetilde{\mathcal{F}}^s_L$. Let $z_i$ in the $z'_i$ orbit with $z_i$ in $l_i$. Then $z_i$ converges to $z$. This shows that any $z$ in $l$ is the limit of a sequence in $l_i$.

Now suppose that $(z_{i_k})$ is a sequence converging to $z$ with $z_{i_k}$ in $L_{i_k}$. Here $p_{i_k}$ is in $L_{i_k}$ and $p$ is in $L$ and hence $L_{i_k}$ converges to $L$ in the leaf space $\mathcal{H}$ of $\widetilde{\mathcal{G}}$. Since $\mathcal{H}$ is Hausdorff then no sequence of points in $L_{i_k}$ converges to a point in another leaf of $\widetilde{\mathcal{G}}$. If follows that $z$ is in $L$. Let

$$V_k \ = \ \widetilde{W}^s(p_{i_k}), \qquad V \ = \ \widetilde{W}^s(p)$$

Then $V_k$ converges to $V$. By lemma 4.1 the leaf space of $\widetilde{\mathcal{F}}^s$ is also Hausdorff. It follows that $z$ is in $V$. Hence $z$ is in $L \cap V = \tau$. It was also proved in [Fe6] that $L \cap V$ is connected and hence $\tau$ is exactly the leaf of $\widetilde{\mathcal{F}}^s_L$ containing $p$.

If $\tau$ is non singular this finishes the proof of the first statement. Suppose then that $\widetilde{W}^s(p)$ is singular. Since the $p_i$ are in the closure of a sector of $\widetilde{W}^s(p)$ then so are the $l_{i_k}$ and hence the $z_{i_k}$. Consequently the same is true of $z$. The boundary of this sector is a line leaf of $\widetilde{W}^s(p)$ and so $z$ is in the corresponding line leaf of $\widetilde{\mathcal{F}}^s_L$, which is $l$. This finishes the proof of the first statement of the lemma.

We now consider the second part of the lemma. By lemma 4.1 the leaves of $\widetilde{\mathcal{F}}^s_E$ are uniform quasigeodesics in $E$ for any $E$ leaf of $\widetilde{\mathcal{G}}$. Let then $b > 0$ so that any line leaf of $\widetilde{\mathcal{F}}^s_E$ is $\leq b$ from the corresponding geodesic in $E$ and likewise for arcs in such leaves. Let $l_i$ be line leaves of $\widetilde{\mathcal{F}}^s_{L_i}$, $l$ its limit in a leaf $L$ of $\widetilde{\mathcal{G}}$ as in the first part of the lemma. Let $s_i$ be the geodesics in $L_i$ corresponding to $l_i$ and let $s$ the geodesic in $L$ corresponding to $l$.

For any $\epsilon > 0$ there is fixed $f(\epsilon) > 0$ so that if two geodesic segments in the hyperbolic plane have length bigger than $3f(\epsilon)$ and corresponding endpoints are less than $2b+2$ from each other, then except for segments of length $f(\epsilon)$ adjacent to the endpoints, then the rest of the segments are less than $\epsilon/3$ from each other.

Let then $z$ in $s$. Given $\epsilon > 0$, find $w', u'$ in $s$ which are $(3f(\epsilon) + 2b + 1)$ distant from $z$. There are $w, u$ in $l$ with

$$d_L(w, w') \ < \ b + \frac{1}{2}, \quad d_L(u, u') \ < \ b + \frac{1}{2}$$

Let $\tau$ be the segment of $l$ between $w, u$. There is a corresponding segment of $\tau_i$ of $l_i$ between points $w_i, u_i$ so that the Hausdorff distance in $\widetilde{M}$ from $\tau$ to $\tau_i$ is $<< 1$. The corresponding geodesic segment $m_i$ from $w_i$ to $u_i$ in $L_i$ is less than $b$ from $\tau_i$ and by choice of $w', u'$ then the midpoint of $m_i$ is less than $\epsilon/3$ from a point $v_i$ in $s_i$. Hence $v_i$ is less than $\epsilon$ from $z$. By adjusting the $\epsilon$ to converge to 0 and the $i$ to increase, one finds $v_i$ in $s_i$ with $v_i$ converging to $z$.

Suppose now that $z_{i_k}$ are in $s_{i_k}$ with $s_{i_k}$ contained in $L_{i_k}$. Suppose the sequence $s_{i_k}$ converges to $z$ in $\widetilde{M}$. The proof is very similar to the above: Fix $\epsilon > 0$. Choose big segments in $s_{i_k}$ centered in $z_{i_k}$. The length is fixed and depends on $\epsilon$. There are geodesic arcs of $L_{i_k}$ with endpoints in the leaves $l_{i_k}$ whose midpoints are very close to $z_{i_k}$. Very close depends on $\epsilon$ and the length above. There are arcs in $l_i$ with these endpoints so that the above arcs converge up to a subsequence to a segment in $l$ by the first part of the lemma. The geodesic arcs above converge to a geodesic arc with endpoints in $l$. Up to subsequence the midpoints of the geodesic arcs (that is, the $z_{i_k}$) converge to a point (this point is $z$) which is close to a point in $s$, closeness depending on $\epsilon$. Now make $\epsilon$ converge to 0 and prove that $z$ is in $l$. This finishes the proof of the lemma. $\square$

At this point it is convenient to do the following: for the remainder of the article we fix a leaf $L$ of $\widetilde{\mathcal{G}}$. The bijection $L \cup \partial_\infty L \to \mathcal{O} \cup \mathcal{U}$ is a homeomorphism. Therefore the action of $\pi_1(M)$ on $\mathcal{O} \cup \mathcal{U}$ induces an action by homeomorphisms on $L \cup \partial_\infty L$ under this identification. This action leaves invariant the foliations $\widetilde{\mathcal{F}}^s_L, \widetilde{\mathcal{F}}^u_L$ – which are the intersections of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ with $L$.

We need one more auxiliary fact.

**Lemma 5.2.** *Let $E$ be a leaf of $\widetilde{\mathcal{G}}$ and $l_1, l_2$ distinct leaves of $\widetilde{\mathcal{F}}^s_E$ or $\widetilde{\mathcal{F}}^u_E$. Then $l_1, l_2$ do not share an ideal point in $\partial_\infty E$.*

*Proof.* Suppose first by way of contradiction that there are $l_1, l_2$ rays in leaves of $\widetilde{\mathcal{F}}^s_E$ for some $E$ in $\widetilde{\mathcal{G}}$ with the same ideal point $a$ in $\partial_\infty E$ and so that $l_1, l_2$ do not share a subray. We can assume that $l_1, l_2$ do not have singularities. Let $u_j, j = 1, 2$ be the starting point of $l_j$. Let $r_j, j = 1, 2$ be a line leaf of $\widetilde{\mathcal{F}}^s_E$ containing $l_j$. Choose points $p_i$ in $l_1$ escaping in $l_1$. As explained before the leaves of $\widetilde{\mathcal{F}}^s_E$ are uniform quasigeodesics in $E$ and hence they are at a bounded distance in $E$ from geodesics in $E$. This implies that there are $q_i$ in $l_2$ so that $q_i$ are a bounded distance from $p_i$ in $E$. Up to taking a subsequence we may assume that $\pi(p_i)$ converges in $M$. Let then $g_i$ in $\pi_1(M)$ with $g_i(p_i)$ converging to $p_0$. For simplicity of explanation we assume that the leaf of $\widetilde{\mathcal{G}}$ containing $p_0$ is the fixed leaf $L$ as above. Let $v_1$ be the line leaf of $\widetilde{\mathcal{F}}^s_L$ containing $p_0$ and which is the limit of the $g_i(r_1)$ as proved in the previous lemma. If $\widetilde{W}^s(p_0)$ is singular then, up to taking a subsequence, we may assume that the $g_i(p_i), g_i(r_i)$ satisfy the requirements of the previous lemma.

Since the distance along $g_i(E)$ from $g_i(p_i)$ to $g_i(q_i)$ is bounded we may assume up to subsequence that $g_i(q_i)$ also converges and let $q_0$ be its limit. It follows that $q_0$ is also in $L$ and let $v_2$ be the line leaf of $\widetilde{\mathcal{F}}^s_L$ containing $q_0$ which is the limit of $g_i(r_2)$. Here the rays

$$g_i(l_1),\ g_i(l_2) \ \text{ in } \ E \quad \text{have the same ideal point} \ \ g_i(a) \ \text{ in } \ \partial_\infty(g_i(E))$$

The line leaves $r_j$ are uniform quasigeodesics in $E$ and a bounded distance from a geodesic $s_j$ in $E$. Hence the geodesics $g_i(s_1), g_i(s_2)$ share an ideal point in $\partial_\infty g_i(E)$. By the second part of the previous lemma $g_i(s_j)$ converges to a geodesic $t_j$ in $L$ with same ideal points as $v_j$ for both $j = 1, 2$. By continuity of geodesics in leaves of $\widetilde{\mathcal{G}}$, it follows that $t_1$ and $t_2$ share an ideal point. Therefore $v_1, v_2$ share an ideal point in $\partial_\infty L$.

We claim that $v_1, v_2$ also share the other ideal point. The line leaves $g_i(r_1), g_i(r_2)$ have big segments from

$$g_i(u_1) \ \text{ to } \ g_i(p_k) \quad \text{ and } \quad g_i(u_2) \ \text{ to } \ g_i(q_k)$$

which are boundedly close to each other. Here $k >> i$ and so $g_i(p_i)$ is in these segments. Also $g_i(p_i)$ converges to $p_0$. The corresponding geodesic arcs between the points above have endpoints which are boundedly close to each other. As explained in the proof of the previous lemma they have middle thirds which are arbitrarily close to each other. The limits of the geodesic arcs are contained in $t_1$ and $t_2$. This shows that $t_1$ and $t_2$ have points in common and therefore are the same geodesic.

Suppose first that $v_1, v_2$ are distinct. The two line leaves $v_1, v_2$ of $\widetilde{\mathcal{F}}^s_L$ have the same two ideal points, which we denote by $a_1, a_2$. The line leaves

$$v_1, \ v_2 \quad \text{bound a region} \quad R \quad \text{in} \quad L$$

For any stable leaf $l$ of $\widetilde{\mathcal{F}}^s_L$ in $R$ then $l$ has ideal points which can only be $a_1, a_2$. But $l$ is a quasigeodesic in $L$. Therefore this leaf is non singular and has ideal points exactly $a_1, a_2$. Now consider a periodic orbit $\alpha$ of $\widetilde{\Phi}$ intersecting $L$ in $R$ very close to $v_1$ so that the unstable leaf $\widetilde{W}^u(\alpha)$ intersects $v_1$. Notice that the set of periodic orbits of $\Phi$ is dense in $M$ when $\Phi$ is transitive as proved by Mosher [Mo1]. In addition if $M$ is atoroidal then $\Phi$ is transitive [Mo1].

We now use that $L \cup \partial_\infty L$ is identified with $\mathcal{O} \cup \mathcal{U}$. Let $g$ in $\pi_1(M)$ non trivial so that $g(\alpha) = \alpha$ and in addition $g$ leaves invariant all components of $\widetilde{W}^s(\alpha) - \alpha$. Under the identifications above then

$$g \quad \text{fixes} \quad a_1 \text{ and } a_2 \quad \text{in} \quad \partial_\infty L$$

Notice that $a_1, a_2$ are the ideal points of $\widetilde{W}^s(\alpha) \cap L$ in $\partial_\infty L$. Assume that $g^n(\widetilde{W}^s(v_1))$ moves away from $\widetilde{W}^s(\alpha)$ when $n$ converges to infinity. Since $v_1$ (line leaf of $\widetilde{\mathcal{F}}^s_L$) has ideal points $a_1, a_2$, it follows that the same happens for all leaves $g^n(\widetilde{W}^s(v_1)) \cap L$. These line leaves are nested in $L$ and they are uniform quasigeodesics in $L$, so they cannot escape compact sets in $L$. Hence they have to limit in a line leaf $v$ of $\widetilde{\mathcal{F}}^s_L$. Since the leaf space of $\widetilde{\mathcal{F}}^s_L$ is Hausdorff, the limit is unique, which implies that $g(v) = v$. The leaf $z$ of $\mathcal{O}^s$ corresponding to $v$ is also invariant under $g$. This produces a point $y$ of $\mathcal{O}$ in $v$ which is invariant under $g$. Let $\beta$ be the orbit of $\widetilde{\Phi}$ with $\Theta(\beta) = y$. But also $g$ leaves invariant the point $x = \Theta(\alpha)$. This shows that there are 2 fixed points in $\mathcal{O}$ under $g$. Then $\pi(\alpha), \pi(\beta)$ are closed orbits of $\widetilde{\Phi}$ which up to powers are freely homotopic to the inverse of each other. Since $\Phi$ is regulating, this is impossible: Notice that $g$ is associated to the negative flow direction in $\alpha$ – as it acts as an expansion in the set of orbits of $\widetilde{W}^u(\alpha)$. The regulating property applied to $\alpha$ implies that $g$ acts freely and in an decreasing fashion on the leaf space $\mathcal{H}$ of $\widetilde{\mathcal{G}}$. The property that $\pi(\beta)$ is freely homotopic to the inverse of $\pi(\alpha)$ implies that $g$ would have to act in a decreasing way on $\mathcal{H}$, contradiction. Notice that the last argument is about the leaf space of $\widetilde{\mathcal{G}}$ and not of $\widetilde{\mathcal{F}}^s$. This contradiction shows that $l_1, l_2$ cannot have the same ideal point in $E$. This finishes the analysis if $v_1, v_2$ are distinct.

If $v_1 = v_2$, then for $i$ big enough we may assume that $p_i$ is very closed to $q_i$. Then one can choose $\alpha$ periodic with $\widetilde{W}^u(\alpha)$ intersecting both $l_1$ and $l_2$. It follows that $\widetilde{W}^s(\alpha) \cap L$ has one endpoint $a$. Then one applies the same arguments as in the case $v_1, v_2$ distinct to produce a contradiction. This finishes the first part of the lemma.

We now prove that if $l_1$ is a ray in a leaf of $\widetilde{\mathcal{F}}^s_L$ and $l_2$ is ray in a leaf of $\widetilde{\mathcal{F}}^u_L$ then they cannot share an ideal point in $\partial_\infty L$. Suppose this is not the case. Apply the same limiting procedure as above to produce a stable line leaf $s_1$ in $\widetilde{\mathcal{F}}^s_L$ and an unstable line leaf $s_2$ in $\widetilde{\mathcal{F}}^u_L$ which share two ideal points. Clearly in this case they cannot be the same leaf and they bound a region $R$ in $L$ with ideal points $a_1, a_2$. Consider a non singular stable leaf $l$ intersecting $s_2$. Then it enters $R$ and cannot intersect the boundary of $R$ (in $L$) again. Therefore it has to limit in either $a_1$ or $a_2$ and share an ideal point with a ray of $s_1$. This is disallowed by the first part of the proof. $\square$

Given these facts the following happens: For any $L$ in $\widetilde{\mathcal{G}}$ and leaf $l$ in $\widetilde{\mathcal{F}}^s_L$ if $l$ is non singular let $l^*$ be the geodesic in $L$ with same ideal points as $l$. If $l$ is a $p$-prong leaf, let $\delta_1, ..., \delta_p$ be the line leaves of $l$ and $\delta_i^*$ be the corresponding geodesics. In this case let $l^*$ be the union of the $\delta_i^*$, which is a $p$-sided ideal polygon in $L$. Let $\widetilde{\mathcal{L}}^s_L$ be the union of such $l^*$ for $l$ in $\widetilde{\mathcal{F}}^s_L$ and similarly define $\widetilde{\mathcal{L}}^u_L$.

By lemma 5.1 it follows that $\widetilde{\mathcal{L}}^s_L, \widetilde{\mathcal{L}}^u_L$ are closed subsets of $L$ and there are geodesic laminations in $L$. The complementary regions of $\widetilde{\mathcal{L}}^s_L$ are exactly those associated to $p$-prong leaves of $\widetilde{\mathcal{F}}^s_L$ this also follows from lemma 5.1 and hence are finite sided ideal polygons. As leaves of $\widetilde{\mathcal{F}}^s_L$ are uniform quasigeodesics (lemma 4.1), then $\widetilde{\mathcal{L}}^s_L$ varies continuously if $L$ varies in $\widetilde{\mathcal{G}}$. This produces a lamination in $M$ which intersects leaves of $\mathcal{G}$ in geodesic laminations. As $\widetilde{\mathcal{F}}^s_L, \widetilde{\mathcal{F}}^u_L$ have no rays which share an ideal point, it follows that $\widetilde{\mathcal{L}}^s_L$ is transverse to $\widetilde{\mathcal{L}}^u_L$. It now follows that for any $p$ in $\partial_\infty L$, then $p$ has a neighborhood system in $L \cup \partial_\infty L$ defined by a sequence of leaves in either $\widetilde{\mathcal{L}}^s_L$ or $\widetilde{\mathcal{L}}^u_L$. herefore the same holds for $\widetilde{\mathcal{F}}^s_L, \widetilde{\mathcal{F}}^u_L$ as these are uniform quasigeodesics.

We now analyse the properties of the action of $\pi_1(M)$ on $\mathcal{U}$.

**Proposition 5.3.** *Let $\mathcal{G}$ be an $\mathbf{R}$-covered foliation with a transverse regulating pseudo-Anosov flow $\Phi$. Let $g$ in $\pi_1(M)$ be a non trivial element. Then one of the following options must happen:*

*$I$ − If $g$ fixes 3 or more points in $\mathcal{U}$, then $g$ does not act freely on $\mathcal{O}$ and has a unique fixed point $x$ in $\mathcal{O}$. Here $g$ is associated to a closed orbit of $\Phi$. In addition $g$ acts by an orientation preserving homeomorphism of $\mathcal{O}$ and $g$ leaves invariant each prong of $\mathcal{O}^s(x), \mathcal{O}^u(x)$ when acting on $\mathcal{O}$. Hence $g$ fixes the ideal points of $\mathcal{O}^s(x), \mathcal{O}^u(x)$ in $\mathcal{U}$ which are even in number. These are the only fixed points of $g$ in $\mathcal{U}$ and they are alternatively repelling and attracting;*

*$II$ − $g$ fixes exactly two points in $\mathcal{U}$. Then, either 1) $g$ acts freely on $\mathcal{O}$ and there is one attracting and one repelling fixed point in $\mathcal{U}$; or 2) $g$ fixes a point $x$ in $\mathcal{O}$ and leaves invariant exactly two prongs of (say) $\mathcal{O}^s(x)$ but not those of $\mathcal{O}^u(x)$ or any other possible prongs of $\mathcal{O}^s(x)$ (or vice versa). Here $g$ reverses orientation in $\mathcal{O}$. The orbit associated to $x$ may be non singular in which case all prongs of $\mathcal{O}^s(x)$ are left invariant and there are 4 fixed points in $\mathcal{U}$ under the square of $g$. The orbit associated to $x$ may be singular. Then the square of $g$ has more than 4 fixed points in $\mathcal{U}$.*

*$III$ − $g$ has no fixed point in $\mathcal{U}$. Then $g$ fixes a single point $x$ in $\mathcal{O}$ and a power of $g$ fixes an even number $\geq 4$ of points in $\mathcal{U}$.*

*Consequently, $g$ always fixes a finite even number of points in $\mathcal{U}$ (it may be zero).*

*Proof.* Since $g$ acts on $\mathcal{O}$ and leaves invariant the foliation $\mathcal{O}^s$, then it acts on the leaf space $\mathcal{H}^s$ of $\mathcal{O}^s$. This is the same as the leaf space of $\widetilde{\mathcal{F}}^s_L$ (under the identification of $\mathcal{O}$ with $L$). Recall that the leaf space of $\mathcal{O}^s$ is Hausdorff. Therefore the leaf space $\mathcal{H}^s$ of $\mathcal{O}^s$ is a topological tree [Fe3]. The same happens for the leaf space of $\mathcal{O}^u$.

Given any $g$ in $\pi_1(M)$ it induces a homeomorphism of this topological tree $\mathcal{H}^s$. $\mathbf{Z}$ actions on such trees are well understood [Ba, Fe3, Ro-St]. There are 2 options:

− $g$ acts freely and has an axis $v$. Elements in the axis are those $z$ in $\mathcal{H}^s$ for which $g(z)$ separates $z$ from $g^2(z)$, or

− $g$ fixes a point in $\mathcal{H}^s$.

Suppose first that $g$ acts freely on $\mathcal{H}^s$. Then $g$ has an axis $v$ for its action on $\mathcal{H}^s$ and consequently an axis for its action on the leaf space of $\widetilde{\mathcal{F}}^s_L$. Let $l$ be a leaf of $\widetilde{\mathcal{F}}^s_L$ in the axis and we may assume that $l$ is non singular. By the axis properties it follows that the leaves

$$\{g^n(l), \quad n \in \mathbf{Z}\}$$

are nested in $L$ and they are uniform quasigeodesics. Since they escape when viewed in the leaf space of $\widetilde{\mathcal{F}}^s_L$, the same is true in $L$. As they are uniform quasigeodesics and nested, then there are unique points $p, q$ in $\partial_\infty L$ so that $g^n(l)$ converges to $p$ if $n$ converges to infinity and to $q$ if $n$ converges to minus infinity. Hence under the identification of $\mathcal{U}$ with $\partial_\infty L$, then $p, q$ are the unique fixed points of (any power of) $g$ in $\mathcal{U}$, where $p$ is attracting and $q$ repelling. In this case the action of $g$ in $\mathcal{O}$ could be orientation preserving or not. This is case II, 1).

From now on in the proof we assume that $g$ has a fixed point in $\mathcal{H}^s$, so there is a leaf $l$ of $\mathcal{H}^s$ with $g(l) = l$. Then the leaf

$$\Theta(l) \ \text{ of } \ \mathcal{O}^s \quad \text{contains a unique} \ \ x \ \text{ in } \ \mathcal{O} \ \text{ with } \ g(x) \ = x$$

If $g$ has no fixed points in $\mathcal{U}$ then it acts as an orientation preserving homeomorphism on $\mathcal{U}$ and hence the same happens for the action on $\mathcal{O}$.

There is a smallest positive integer $i_0$ so that $h = g^{i_0}$ leaves invariant all prongs of $\mathcal{O}^s(x), \mathcal{O}^u(x)$. If there are $2n$ such prongs, each generates an ideal point of $L$ and also a point of $\mathcal{U}$. By lemma 5.2 any two distinct prongs have different ideal points in $\mathcal{U}$. Hence $h$ has at least $2n$ fixed points in $\mathcal{U}$. Let $\alpha$ be the flow line of $\widetilde{\Phi}$ with $\Theta(\alpha) = x$. Without loss of generality assume that the prongs above are circularly ordered with corresponding ideal points

$$a_1, b_1, ..., a_n, b_n \ \text{ in } \ \mathcal{U} \ \text{ where } \quad \partial\mathcal{O}^s(x) = \{a_1, a_2, ..., a_n\}, \ \ \partial\mathcal{O}^u(x) = \{b_1, b_2, ..., b_n\}.$$

Suppose that $g$ is associated to the positive flow direction in $\alpha$. Fix a prong $\tau$ of $\mathcal{O}^s(x)$ and let $I$ be the maximal interval of $\mathcal{U} - \partial\mathcal{O}^u(x)$ containing the ideal point of $\tau$. Let now $\mu$ be an arbitrary unstable leaf of $\mathcal{O}^u$ intersecting $\tau$. Then as $\mu$ gets closer to prongs of $\mathcal{O}^u(x)$, the ideal points of $\mu$ approach the endpoints of $I$. The action of $h$ on $\tau$ is as follows: $h$ fixes $x$ and for a leaf $\mu$ as above then $h$ takes it to a leaf farther away from $x$. This is because in $\widetilde{M}$ the flow lines along stable leaves move closer in forward time. Notice that $h$ acts on $\widetilde{M}$ as an isometry. This isometry takes a flow line in $\widetilde{W}^s(\alpha)$ to one which is at the same distance from $\alpha$, but farther away from $\alpha$ than $\beta$ is $-$ as $\beta$ is getting closer to $\alpha$. Flowing back to the original position it means that the image is farther from the orbit $\alpha$ thatn $\beta$ is. It follows that $h$ acts as an expansion in $\tau$ with a single fixed point in $x$. Given $\mu$ as above then $h^n(\mu) \cap \tau$ escapes in $\tau$ as $n$ converges to infinity. These also form a nested collection of leaves. If the sequence $h^n(\mu)$ does not escape compact sets in $\mathcal{O}$, then it limits in a collection

$$\mathcal{W} = \{W_i, i \in C\}$$

of leaves of $\mathcal{O}^u$, where $C$ is an interval in $\mathbf{Z}$ either finite or all of $\mathbf{Z}$ [Fe6]. In addition $h$ leaves invariant $\mathcal{W}$. If $\mathcal{W}$ is not finite, then in particular it is not a single point and then the leaf space of $\mathcal{O}^s$ is not Hausdorff, which is impossible as seen previously. If on the other hand $\mathcal{W}$ is a single leaf $W$, then $h(W) = W$ and there is a single periodic point $z$ in $W$ with $h(z) = z$. Then $h$ fixes $x$ and $z$ and this is also impossible as seen above.

It follows that $h^n(\mu)$ escapes compact sets in $\mathcal{O}$ and as seen in the free action case, they can only limit in a single point of $\mathcal{U}$, which corresponds to the ideal point $p$ of $\tau$. This shows that $h$ acts as a contraction in $I$ with fixed point $p$. Hence the points $a_i, 1 \leq i \leq n$ are attracting fixed points of $h$ in $\mathcal{U}$. Using $h^{-1}$ one shows that the $b_i, 1 \leq i \leq n$ are repelling fixed points and these are the only fixed points of $h$ in $\mathcal{U}$. Hence $h$ fixes exactly $2n$ points in $\mathcal{U}$, where $n \geq 2$.

We now return to $g$. If $g$ is orientation reversing on $\mathcal{U}$, then so is the action on $\mathcal{O}$. In this case there are exactly 2 fixed points of $g$ in $\mathcal{U}$. The square of $g$ is now orientation preserving on $\mathcal{U}$ and it has fixed points. In particular any fixed point of $g^{2i}$ is a fixed point of $g^2$. It follows that $h$ is equal to $g^2$ and this is case II, 2).

Suppose finally that $g$ is orientation preserving on $\mathcal{U}$. Since $h = g^{i_0}$ has fixed points in $\mathcal{U}$, then either $g$ has no fixed points in in $\mathcal{U}$ or $g$ has exactly the same fixed points in $\mathcal{U}$ as $h$ does. In the second case $h$ is equal to $g$ and $g$ has exactly $2n$ fixed points in $\mathcal{U}$, which are alternatively attracting and contracting. This is case I). In the first case $g$ acts essentially as a rotation in $\mathcal{U}$ and $\mathcal{O}$. This is case III).

This finishes the proof of the proposition. □

## 6 Construction of the conjugacy

Let now $\Phi, \Psi$ be two pseudo-Anosov flows transverse to the $\mathbf{R}$-covered foliation $\mathcal{G}$ and both regulating for $\mathcal{G}$. We want to show that $\Phi$ and $\Psi$ are topologically conjugate. Let $\mathcal{O}$ be the orbit space of $\widetilde{\Phi}$ and $\mathcal{T}$ be the orbit space of $\widetilde{\Psi}$. We will construct a $\pi_1(M)$-equivariant homeomorphism from $\mathcal{O}$ to $\mathcal{T}$. We first associate to each periodic orbit of $\Phi$ a unique periodic orbit of $\Psi$. Let

$$\Theta_1 : \widetilde{M} \rightarrow \mathcal{O} \quad \text{and} \quad \Theta_2 : \widetilde{M} \rightarrow \mathcal{T}$$

be the corresponding orbit space projection maps. Let $\mathcal{O}^s, \mathcal{O}^u$ be the projection stable and unstable foliations of $\widetilde{\Phi}$ to $\mathcal{O}$ and $\mathcal{T}^s, \mathcal{T}^u$ the corresponding objects for $\widetilde{\Psi}$. Recall that $\pi : \widetilde{M} \rightarrow M$ is the universal covering map.

One main property to note here is that the universal circle $\mathcal{U}$ depends only on $\mathcal{G}$ and not on $\Phi$ or $\Psi$. The same is true for the action of $\pi_1(M)$ on $\mathcal{U}$. Before we prove the theorem, we first prove a preliminary property:

**Lemma 6.1.** *Let $\alpha$ be an orbit of $\widetilde{\Phi}$ so that $\pi(\alpha)$ is a closed orbit of $\Phi$. Let $g$ be the element of $\pi_1(M)$ associated to the closed orbit $\pi(\alpha)$. Then there is a unique closed orbit $\beta$ of $\widetilde{\Psi}$ so that $\pi(\beta)$ is periodic and associated to $g$, that is, $\pi(\beta)$ is freely homotopic to $\pi(\alpha)$.*

*Proof.* Let $x = \Theta_1(\alpha)$ and $g$ non trivial in $\pi_1(M)$ with $g(x) = x$ and indivisible with respect to this property. Suppose that $g$ is associated to the forward flow direction of $\pi(\alpha)$. Let $h$ be the smallest power of $g$ so that $h$ leaves invariant all prongs of $\mathcal{O}^s(x), \mathcal{O}^u(x)$. Proposition 5.3 shows that $h$ has $2n$ fixed points in $\mathcal{U}$, with $n \geq 2$. This is case I). Now apply this proposition to $h$ and $\Psi$. Since $h$ has $2n$ fixed points in $\mathcal{U}$ and $n \geq 2$, proposition 5.3 implies that there is a unique $y$ in $\mathcal{T}$ with $h(y) = y$. Let

$$\beta \ \text{be the orbit of} \ \widetilde{\Psi} \ \text{with} \ \Theta_2(\beta) = y, \ \text{so} \ h(\beta) = \beta$$

If $g$ acts freely on $\mathcal{T}$ then the analysis of proposition 5.3 shows that $h$ can have only 2 fixed points in $\mathcal{U}$, impossible. It follows that $g$ cannot act freely on $\mathcal{T}$ and therefore the only fixed point of $g$ in $\mathcal{T}$ is $y$ − as it is fixed under a power of $g$. This implies that $g(\beta) = \beta$ and consequently $\pi(\alpha)$ is freely homotopic to a power of $\pi(\beta)$. Reversing the roles of $\alpha$ and $\beta$ implies that $\pi(\alpha)$ and $\pi(\beta)$ are freely homotopic to each other or their inverses. The action of $h$ on $\mathcal{U}$ shows that the first option is the one that happens − this is because they both have attracting fixed points in $\mathcal{U}$ in the same points. This finishes the proof of the lemma. □

This defines a map from the periodic points of $\mathcal{O}$ to the periodic points of $\mathcal{T}$. Notice that in the lemma above $\partial\mathcal{O}^s(x) = \partial\mathcal{T}^s(y)$ as points in $\mathcal{U}$ and similarly for $\mathcal{O}^u(x), \mathcal{T}^u(y)$.

**Theorem 6.2.** *Let $\Phi, \Psi$ be pseudo-Anosov flows, which are transverse and regulating for an* $\mathbf{R}$-*covered foliation $\mathcal{G}$. Then $\Phi, \Psi$ are topologically conjugate.*

*Proof.* We first define a map $f$ from $\mathcal{O}$ to $\mathcal{T}$ which extends the correspondence between periodic points obtained previously. Given $x$ in $\mathcal{O}$, we will let $y$ be the unique point of $\mathcal{T}$ with

$$\partial \mathcal{T}^s(y) \ = \ \partial \mathcal{O}^s(x), \quad \partial \mathcal{T}^u(y) \ = \ \partial \mathcal{O}^u(x)$$

If $x$ is periodic, this was constructed in the previous lemma. The previous lemma shows that there is always such a $y$. Lemma 5.2 shows that $y$ is uniquely defined. This is because the set $\partial \mathcal{T}^s(y)$ determines the stable leaf $\mathcal{T}^s(y)$ and likewise for $\mathcal{T}^u(y)$. Hence $y$ is uniquely defined. If $x$ is not periodic let $x_n$ in $\mathcal{O}$ which are periodic and converging to $x$. We may assume that no $x_n$ is singular since the singular orbits form a discrete subset of $\mathcal{O}$. We can also assume that $(\mathcal{O}^s(x_n))$ forms a nested sequence, and so does $(\mathcal{O}^u(x_n))$. Let $p_n, q_n$ points in $\mathcal{U}$ with

$$\partial \mathcal{O}^s(x_n) \ = \ \{p_n, q_n\} \quad \text{and let} \quad \{p, q\} \ = \ \partial \mathcal{O}^s(x)$$

Then up to renaming we can assume that $p_n$ converges to $p$ in $\mathcal{U}$ and $q_n$ converges to $q$ in $\mathcal{U}$. Let $y_n$ in $\mathcal{T}$ periodic with $\partial \mathcal{T}^s(y_n) = \{p_n, q_n\}$. Notice that the $l_n = \mathcal{T}^s(y_n)$ are leaves of $\mathcal{T}^s$, which are nested in $\mathcal{T}$. By the identification of $L$ with $\mathcal{O}$, then the $l_n$ are associated to uniform quasigeodesics in $L$ which have ideal points which converge to distinct points in $\partial_\infty L$ (associated to $p, q$ in $\mathcal{U}$). Therefore these quasigeodesics converge to a single quasigeodesic in $L$ and so

$$\mathcal{T}^s(y_n) \quad \text{converges to a leaf} \quad l \quad \text{of} \quad \mathcal{T}^s$$

Similarly $\mathcal{T}^u(y_n)$ converges to a leaf $s$ of $\mathcal{T}^u$. For all $n$, the pairs $\partial \mathcal{O}^s(x_n)$, $\partial \mathcal{O}^u(x_n)$ link each other in $\mathcal{U}$, so the same happens for $\partial \mathcal{T}^s(y_n), \partial \mathcal{T}^u(y_n)$. It follows that the ideal points of $l$, $s$ link each other in $\mathcal{U}$ otherwise we would have a leaf of $\mathcal{T}^s$ sharing an ideal point with a leaf of $\mathcal{T}^u$ − which is disallowed by lemma 5.2. Therefore

$$y_n \ = \ \mathcal{T}^s(y_n) \cap \mathcal{T}^u(y_n)$$

converges to a point $y$ in $\mathcal{T}$. Clearly $\partial \mathcal{T}^s(y)$ contains $\partial \mathcal{O}^s(x)$ and similarly $\partial \mathcal{T}^u(y)$ contains $\partial \mathcal{O}^u(x)$. If $x$ is a singular orbit, one could apply the inverse process to produce $x'$ in $\mathcal{O}$, $x'$ singular so that $\partial \mathcal{O}^s(x') = \partial \mathcal{T}^s(y)$. But then $\partial \mathcal{O}^s(x')$ contains $\partial \mathcal{O}^s(x)$ and $x$ is non singular. This is disallowed by the lemma. Therefore $y$ is non singular and hence

$$\partial \mathcal{T}^s(y) \ = \ \partial \mathcal{O}^s(x), \quad \partial \mathcal{T}^u(y) \ = \ \partial \mathcal{O}^u(x)$$

In addition, since no two stable leaves of $\mathcal{O}^s$ can share an ideal point, then $y$ is well defined.

This map $f : \mathcal{O} \to \mathcal{T}$ is well defined. It is also injective. If $f(x_1) = f(x_2)$ then $\partial \mathcal{O}^s(x_1) = \partial \mathcal{O}^s(x_2)$ and $\partial \mathcal{O}^u(x_1) = \partial \mathcal{O}^u(x_2)$. By lemma 5.2, the first fact implies that $\mathcal{O}^s(x_1) = \mathcal{O}^s(x_2)$ and the second fact implies that $\mathcal{O}^u(x_1) = \mathcal{O}^u(x_2)$. Therefore their intersection is $x_1 = x_2$ and the map $f$ is injective. In addition, the map $f$ clearly has an inverse by doing the same procedure from $\Psi$ to $\Phi$. Therefore $f$ is a bijection.

We claim that $f$ is continuous and by symmetry, then the inverse will also be continuous. Let then $x$ in $\mathcal{O}$ and $(x_n)$ a sequence in $\mathcal{O}$ converging to $x$. Assume first that $x$ is non singular. Then

$$\mathcal{O}^s(x_n) \ \text{conveges to} \ \mathcal{O}^s(x) \ \text{and} \ \partial \mathcal{O}^s(x_n) \ \text{converges to} \ \partial \mathcal{O}^s(x) \ \text{in} \ \mathcal{U}$$

Hence $\partial \mathcal{T}^s(f(x_n))$ converges to $\partial \mathcal{T}^s(f(x))$ and similarly for $\partial \mathcal{T}^u(f(x_n))$. This shows that $f(x_n)$ converges to $f(x)$ in $\mathcal{T}$.

Suppose finally that $x$ is singular. Up to subsequence we may assume that $(x_n)$ are all in a sector of $\mathcal{O}^s(x)$ bounded by the line leaf $l$ (contained in $\mathcal{O}^s(x)$). Then $\mathcal{O}^s(x_n)$ converges to $l$ and $\partial \mathcal{O}^s(x_n)$ converges to $\partial l$ in $\mathcal{U}$. It follows that

$$\partial \mathcal{T}^s(f(x_n)) \;\; \text{converges to} \;\; \partial l \; - \; \text{a subset of} \;\; \mathcal{U}$$

which is contained in $\partial \mathcal{T}^s(f(x))$. The same happens if $x_n$ are in $\mathcal{O}^s(x)$, that is, $\partial \mathcal{T}^s(f(x_n))$ are contained in $\partial \mathcal{T}^s(f(x))$. This shows that $\partial \mathcal{T}^s(f(x_n))$ only accumulates in $\partial \mathcal{T}^s(f(x))$. The same is true for $\partial \mathcal{T}^u(f(x_n))$, which only accumulates in $\partial \mathcal{T}^u(f(x))$. Then

$$f(x_n) \;\; = \;\; \mathcal{T}^s(f(x_n)) \; \cap \; \mathcal{T}^u(f(x_n))$$

only accumulates in

$$f(x) \;\; = \;\; \mathcal{T}^s(f(x)) \; \cap \; \mathcal{T}^u(f(x))$$

This shows that $f(x_n)$ has to converge to $f(x)$. This shows that $f$ is a homeorphism from $\mathcal{O}$ to $\mathcal{T}$.

In addition $f$ is $\pi_1(M)$ equivariant: If $g$ is in $\pi_1(M)$ and $x$ is in $\mathcal{O}$, then

$$g(\mathcal{O}^s(x)) \; = \; \mathcal{O}^s(g(x)), \;\;\; g(\mathcal{O}^u(x)) \; = \; \mathcal{O}^u(g(x))$$

have ideal points

$$\partial \mathcal{O}^s(g(x)) \quad \text{and} \quad \partial \mathcal{O}^u(g(x))$$

respectively. Hence these are also the ideal points of

$$\mathcal{T}^s(f(g(x))), \;\;\; \mathcal{T}^u(f(g(x)))$$

In addition

$$\partial \mathcal{T}^s(f(x)) \; = \; \partial \mathcal{O}^s(x) \qquad \text{and} \qquad \partial g(\mathcal{O}^s(f(x))) \; = \; \partial \mathcal{T}^s(g(f(x)))$$

Hence they are the same as $\mathcal{T}^s(f(g(x))$. Since this is also true for the unstable foliations, it follows that

$$f(g(x)) \; = \; g(f(x)) \qquad - \qquad \pi_1(M) \text{ equivariance.}$$

We now finish the proof of topological conjugacy between $\Phi$ and $\Psi$. We define a map $h : \widetilde{M} \to \widetilde{M}$ as follows. Given $p$ in $\widetilde{M}$, then $p$ is in a leaf $L$ of $\widetilde{\mathcal{G}}$. Define

$$h(p) \;\; = \;\; \widetilde{\Psi}_{\mathbf{R}}(f(\Theta(p))) \cap L,$$

here $\Theta(p)$ is in $\mathcal{O}$ and $f(\Theta(p))$ is in $\mathcal{T}$. Essentially we look at the orbit $\alpha = \widetilde{\Phi}_{\mathbf{R}}(p)$ of the flow $\widetilde{\Phi}$ through $p$ and consider the corresponding orbit of $\widetilde{\Psi}$ under the map $f$: that is the orbit $\widetilde{\Psi}_{\mathbf{R}}(f(\Theta(p)))$ of $\widetilde{\Psi}$. Then we intersect this orbit of $\widetilde{\Psi}$ with $L$. This map $h$ preserves the leaves of $\widetilde{\mathcal{G}}$ – not just the foliation $\widetilde{\mathcal{G}}$, but the leaves themselves. In addition $h$ sends orbits of $\widetilde{\Phi}$ to orbits of $\widetilde{\Psi}$. The map $h$ is clearly continuous and hence defines a homeomorphism of $\widetilde{M}$. From the equivariance of $f$ it follows that $h$ is also equivariant, that is for any $g$ and $p$ in $\widetilde{M}$, then $h(g(p)) = g(h(p))$. Therefore $h$ induces a homeomorphism of $M$, which sends orbits of $\Phi$ to orbits of $\Psi$. hence $\Phi$ and $\Psi$ are topologically conjugate. This finishes the proof of theorem 6.2. $\square$

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